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[1] H. A. BETHE, Ann. Rev. Nucl. Sci. **21**, 93 [1971].
[2] C. BLOCH and J. HOROWITZ, Nucl. Phys. **8**, 91 [1958].
[3] M. B. JOHNSON and M. BARANGER, Ann. Phys. New York **62**, 172 [1972].
[4] T. H. SCHUCAN and H. A. WEIDENMÜLLER, Ann. Phys. New York **73**, 108 [1972].
[5] D. M. BRINK and A. WEIGUNY, Nucl. Phys. **A 120**, 59 [1968].
[6] J. HADERMANN, Nucl. Phys. **A 175**, 641 [1971].
[7] A. J. ALVES and E. M. SILVA, Nucl. Phys. **A 181**, 669 [1972]
[8] D. SCHÜTTE and K. BLEULER, Nucl. Phys. **A 119**, 221 [1968].
[9] D. AGASSI, Nucl. Phys. **A 116**, 49 [1968].
[10] Für die Definition der Operatoren siehe Ref. [8], deren Notation hier und im folgenden gebraucht wird.
[11] Bei der Definition $< \Phi \mid = \ll 0 \parallel \exp(z_1 \tau_+ + z_2 A_+{}^+ + z_3 A_-)$ ist zwar die Symmetrie optimal berücksichtigt und $\mathcal{L}$ von 2. Ordnung, aber der Zustand hat keine feste Teilchenanzahl.
[12] B. JANCOVICI and D. H. SCHIFF, Nucl. Phys. **58**, 678 [1964].
[13] J. DA PROVIDENCIA et al., Nucl. Phys. **A 170**, 129 [1971].
[14] K. BLEULER, A. FRIEDERICH, and D. SCHÜTTE, Nucl. Phys. **A 126**, 628 [1969].
[15] K. BLEULER, W. GERLING, and D. SCHÜTTE, unpubliziert, Bonn 1969.
[16] Es hat sich gezeigt, daß die in Ref. [6] angegebene symmetriemischende Approximation auf ein nichtlineares Gleichungssystem führt. Seine Lösungen ergeben einmal die in diesem Abschnitt angegebene Näherung und zum andern Energien, die in den Parametern V und G äußerst unstetig sind.

Coordinate-Independent Green's Functions for the Chiral-Invariant Pion Model

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To KONRAD BLEULER *on the occasion of his 60th birthday*

A coordinate-independent formulation of the chiral-symmetric pion theory is developed using a dreibein field which introduces a redundant gauge field into the field equations. There is a particular gauge, where the field variables transform linearly under chiral transformations and the field equations reduce to those of the Sugawara-model. Quantization of Sugawara's field equations is obtained in terms of Green's functions. The field equations for the coordinate-independent Green's functions are translated into equations for coordinate Green's functions by introduction of the coordinates as auxiliary variables. The latter field equations incorporate the additional Feynman graphs discovered before.

I. Introduction

The success of the chiral-invariant nonlinear pion model in the tree approximation has motivated a number of authors to study the perturbation theory of this model in general. Two major obstacles have to be overcome, before the perturbation series can be given a well defined meaning. The more severe one arises from the fact that the Lagrangian is of the nonpolynomial type and, therefore, nonrenormalizable. Recently, some progress in the study of nonpolynomial Lagrangians has been made by the use of superpropagators. In context with chiral-invariant pion Lagrangians superpropagators have been discussed by LEHMANN and TRUTE [1] and KRAUSE and SCHEUNERT [2]. The superpropagator method requires localizability. LEHMANN and TRUTE [1] have shown that this condition determines the pion field coordinates uniquely. Here the second difficulty, we have mentioned, shows up. The perturbation series should, of course, be independent of the choice of coordinates and moreover, it should be chiral-invariant.

In the following we develop a formulation of the chiral-symmetric pion theory which is independent of the coordinates, while the chiral transformations are reduced to linear transformations of the field variables. The problem to find a coordinate-independent representation is quite similar to that of giving a gauge-independent formulation of electrodynamics. All we have to do, is to introduce a dreibein field and to take components of the field variables with respect to it. In Section II we show that the field equations for the pion coordinates are transformed

into field equations for an isovector current and a redundant gauge field reflecting the freedom to choose a particular dreibein field. The chiral transformations are in general accompanied by a change of gauge. But there is a particular gauge in which the field variables transform linearly under chiral transformations. The field equations then coincide with those of the SUGAWARA model [3]. This matter is discussed in Section III. Finally, in Section IV, we treat the quantization of Sugawara's field equations. This is done in terms of Green's functions which we write in the compact notation introduced by MANDELSTAM [4]. The coordinate-independent and chiral-symmetric Green's functions are transformed into the pion field Green's functions by the introduction of coordinates, serving as auxiliary variables quite similar as the potentials do in electromagnetic and Yang-Mills field theory. The perturbation series for the coordinate Green's functions contains the additional Feynman graphs, discovered by many authors [5–7].

II. Coordinate-Independent Representation

The chiral-symmetric pion Lagrangian is usually written in the geometric form [8]

$$\mathcal{L} = \tfrac{1}{2}\, g_{ij}(\pi)\, \partial_\mu \pi^i \partial^\mu \pi^j , \tag{2.1}$$

where $\pi^i(x)$ $(i = 1, 2, 3)$ are the coordinates of the pion field and g_{ij} is the metric tensor of the Riemannian space S_3. The curvature tensor of S_3 can be expressed in terms of the metric tensor

$$\begin{aligned} R_{ijkl} &= \Gamma_{i,\,jl|k} - \Gamma_{i,\,jk|l} + \Gamma^m_{jl}\ \Gamma_{i,\,mk} - \Gamma^m_{jk}\, \Gamma_{i,\,ml} \\ &= (1/f^2)\,(g_{ik}\, g_{jl} - g_{il}\, g_{jk}), \end{aligned} \tag{2.2}$$

where f is the radius of the sphere S_3, $\Gamma_{i,jl}$ are Christoffel symbols of the first kind and $|\,k$ denotes $\partial\,|\,\partial \pi^k$.

We now introduce a dreibein field $e_i^{\,a}(\pi)$ $(a = 1, 2, 3)$ on S_3. Here and in the following we use subscripts or superscripts from the middle of the Latin alphabet to denote the pion coordinates. Contravariant vectors are denoted by superscripts, covariant vectors by subscripts. Superscripts from the beginning of the Latin alphabet denote the vectors of the dreibein field. There is no distinction between contravariant and covariant components in this case. The contravariant dreibein vectors are defined by

$$e^{ai}(\pi) = g^{ij}(\pi)\, e_j^{\,a}(\pi). \tag{2.3}$$

A dreibein field is defined by the properties

$$e_i^{\,a}(\pi)\, e_j^{\,a}(\pi) = g_{ij}(\pi) \tag{2.4a}$$

$$e_i^{\,a}(\pi)\, e^{bi}(\pi) = \delta^{ab}. \tag{2.4b}$$

Under coordinate transformations it transforms as a vector field

$$\bar{e}_j^{\,a}(\bar{\pi})\, \partial \bar{\pi}^j / \partial \pi^i = e_i^{\,a}(\pi), \quad \bar{\pi}^i = \bar{\pi}^i(\pi). \tag{2.5}$$

The components of a vector field $V^i(\pi)$ on S_3 with respect to the dreibein field are then scalars,

$$V^a(\pi) \equiv e_i^{\,a}(\pi)\, V^i(\pi) = \bar{e}_i^{\,a}(\bar{\pi})\, \bar{V}^i(\bar{\pi}). \tag{2.6}$$

A dreibein field may be defined as solution of the differential equations

$$e^a_{i\|j} = e^a_{i|j} - \Gamma^k_{ij}\, e^a_k = C_j^{\,ab}\, e^b_i. \tag{2.7}$$

The symbol "$\|$" denotes covariant differentiation and Γ^k_{ij} are Christoffel symbols of the second kind. The vector field $C_j^{\,ab}$ is obtained by taking dreibein components of the tensor $e^a_{i\|j}$ with respect to the index i. We may choose any vector field that satisfies the two following conditions: Firstly, it follows by differentiation of (2.4 b) that the matrix $C_j^{\,ab}$ is skew symmetric

$$C_j^{\,ab} + C_j^{\,ba} = 0. \tag{2.8a}$$

Secondly, the integrability of the system (2.7) has to be secured. The condition

$$e^a_{i|j|k} - e^a_{i|k|j} = 0$$

requires

$$\begin{aligned} C^{ab}_{j|k} - C^{ab}_{k|j} + C^{ac}_j\, C^{cb}_k - C^{ac}_k\, C^{cb}_j &= R_{jkmn}\, e^{am}\, e^{bn} \\ &= \frac{1}{f^2}\,(e^a_j\, e^b_k - e^a_k\, e^b_j), \end{aligned} \tag{2.8b}$$

where we have used (2.2) to obtain the last term.

Since any skew symmetric matrix can be written in the form

$$C_i^{\,ab} = \varepsilon^{abc}\, C_i^{\,c}, \tag{2.9}$$

the dreibein field $e_i^{\,a}(\pi)$ is actually determined in terms of a second vector field $C_i^{\,a}(\pi)$ on S_3. The two vector fields $e_i^{\,a}(\pi)$ and $C_i^{\,a}(\pi)$ suggest the introduction of two coordinate-independent Lorentz vector fields:

$$V_\mu^{\,a}(x) = e_i^{\,a}\big(\pi(x)\big)\, \partial_\mu \pi^i(x), \tag{2.10}$$

$$C_\mu^{\,a}(x) = C_i^{\,a}\big(\pi(x)\big)\, \partial_\mu \pi^i(x). \tag{2.11}$$

The integrability condition (2.8 b) can easily be written as field equation in space time

$$\partial_\nu \boldsymbol{C}_\mu - \partial_\mu \boldsymbol{C}_\nu - \boldsymbol{C}_\mu \times \boldsymbol{C}_\nu = (1/f^2)\, \boldsymbol{V}_\mu \times \boldsymbol{V}_\nu. \tag{2.12a}$$

We use $\mathbf{V}_\mu$ and $\mathbf{C}_\mu$ as abbreviation for $V_\mu{}^a$ and $C_\mu{}^a$ $(a=1, 2, 3)$. The cross product equally refers to dreibein indices. A second field equation can be obtained from (2.7). If we consider the rotation

$$e^a_{i|j} - e^a_{j|i} = C_j{}^{ab}\, e_i{}^b - C_i{}^{ab}\, e_j{}^b = \varepsilon^{abc}\, e_i{}^b\, C_j{}^c - \varepsilon^{abc}\, e_j{}^b\, C_i{}^c, \qquad (2.13)$$

we are led to the field equation

$$(\partial_\nu + \mathbf{C}_\nu \times)\, \mathbf{V}_\mu - (\partial_\mu + \mathbf{C}_\mu \times)\, \mathbf{V}_\nu = 0. \qquad (2.12b)$$

Finally a third equation results from the field equation for the pion coordinates corresponding to the Lagrangian (2.1),

$$\Box\, \pi^i + \Gamma^i_{jk} \partial_\mu \pi^j\, \partial^\mu \pi^k = 0\,. \qquad (2.14)$$

In fact, taking the divergence of the vector field $\mathbf{V}_\mu(x)$ we obtain

$$\partial^\mu V_\mu{}^a = \partial^\mu (e_i{}^a\, \partial_\mu \pi^i) = e_i{}^a\, (\Box\, \pi^i + \Gamma^i_{jk}\, \partial_\mu \pi^j\, \partial^\mu \pi^k) + \varepsilon^{abc}\, V_\mu{}^b\, C^{c\mu}.$$

Hence

$$(\partial_\mu + \mathbf{C}_\mu \times)\, \mathbf{V}^\mu = 0\,. \qquad (2.12c)$$

The field Eqs. (2.12) represent a coordinate-independent representation of the pion field theory. They are covariant with respect to the gauge transformations

$$V'^a_\mu(x) = U^{ab}(x)\, V^b_\mu(x)\,, \qquad (2.15a)$$

$$C'_\mu(x) = U\, C_\mu\, U^{-1} + (\partial_\mu U)\, U^{-1}, \qquad (2.15b)$$

where $U(x)$ is an orthogonal matrix field and $(C_\mu)^{ab} = C_i{}^{ab}\, \partial_\mu \pi^i$ acts as a gauge field. The gauge covariance reflects the fact that the dreibein field $e_i{}^a(\pi)$ is determined only up to an orthogonal transformation by the conditions (2.4). If the dreibein field $e_i{}^a$ satisfies (2.7), the transformed field

$$e'^a_i(\pi) = U^{ab}(\pi)\, e_i{}^b(\pi), \qquad (2.16)$$

where $U(\pi)$ is an orthogonal matrix dependent on the pion coordinates, obeys

$$e'^a_{i|j} = \Gamma^k_{ij}\, e'^a_k + C'^{ab}_j\, e'^b_i \qquad (2.17a)$$

with

$$C'_j = U\, C_j\, U^{-1} + U|_j\, U^{-1}\,. \qquad (2.17b)$$

III. Chiral Invariance

The space S_3 can be identified with the homogeneous space $SU(2) \times SU(2) \,|\, SU(2)$. Hence the group $SU(2) \times SU(2)$ may be represented by isometric transformations of the metric g_{ij} [9] which satisfy

$$g'_{ij}(\pi') = g_{mn}(\pi)\, \frac{\partial \pi^m}{\partial \pi'^i}\, \frac{\partial \pi^n}{\partial \pi'^j} = g_{ij}(\pi'). \qquad (3.1)$$

We consider e. g. the metric in Riemannian normal coordinates,

$$g_{ij}(\pi) = \frac{\sin^2 \sqrt{\boldsymbol{\pi}^2/f^2}}{\boldsymbol{\pi}^2/f^2}\, \delta_{ij} + \left(1 - \frac{\sin^2 \sqrt{\boldsymbol{\pi}^2/f^2}}{\boldsymbol{\pi}^2/f^2}\right) \frac{\pi_i\, \pi_j}{\boldsymbol{\pi}^2} \qquad (3.2)$$

where $\boldsymbol{\pi}$ denotes the coordinates π^i $(i=1, 2, 3)$. The group $SU(2) \times SU(2)$ is represented by the infinitesimal transformations

Isospin transformations:

$$\pi'^i = \pi^i + \delta\pi^i = \pi^i + \varepsilon^i{}_{aj}\, \lambda^a(V)\, \pi^j\,, \qquad (3.2a)$$

Chiral transformations:

$$\pi'^i = \pi^i + \delta\pi^i = \pi^i + t^i_a(\pi)\, \lambda^a(A)\,. \qquad (3.2b)$$

$\lambda^a(V)$ and $\lambda^a(A)$ are the group parameters. The tangent vectors $t_a{}^i(\pi)$ to the chiral transformations are given in the coordinate system choosen by

$$t_a{}^i(\pi) = \sqrt{\boldsymbol{\pi}^2}\, \operatorname{ctg} \sqrt{\frac{\boldsymbol{\pi}^2}{f^2}}\, \delta_a{}^i + f\left(1 - \sqrt{\frac{\boldsymbol{\pi}^2}{f^2}}\, \operatorname{ctg} \sqrt{\frac{\boldsymbol{\pi}^2}{f^2}}\right) \frac{\pi^i\, \pi_a}{\boldsymbol{\pi}^2}\,. \qquad (3.3)$$

It may easily be checked that the conditions (3.1) are fulfilled.

Because of (3.1) isometric transformations induce orthogonal transformations of the dreibein field as (2.16), where the matrix U in general depends on the coordinates. Can we choose the dreibein field such that the chiral transformations reduce to linear transformations? Further information on the structure of dreibein fields can be extracted from Eqs. (2.8 b) and (2.13). Addition or subtraction of these equations yields

$$[C_i{}^a \pm (1/f)\, e_i{}^a]_{|j} - [C_j{}^a \pm (1/f)\, e_j{}^a]_{|i} = \varepsilon^{abc}[C_i{}^b \pm (1/f)\, e_i{}^b]\, [C_j{}^c \pm (1/f)\, e_j{}^c]. \qquad (3.4)$$

The general solution of (3.4) is

$$\varepsilon^{abc}[C_i{}^c \pm (1/f)\, e_i{}^c] = -\,(U^{-1}\, U_{|i})^{ab}\,, \qquad (3.5)$$

where $U(\pi)$ is an arbitrary orthogonal matrix. By comparison with (2.17) we see that any dreibein field can be reached by orthogonal transformation with $|U| = +1$ from the particular dreibein fields $e_i{}^a$ defined by

$${}^{(\pm)}e^a_{i|j} = \Gamma^k_{ij}\, {}^{(\pm)}e_k{}^a \mp (1/f)\, \varepsilon^{abc}\, {}^{(\pm)}e_i{}^b\, {}^{(\pm)}e_j{}^c\,. \qquad (3.6)$$

How do the fields ${}^{(\pm)}e_i^a$ transform under chiral transformations? For a general coordinate transformation we have according to (2.5)

$$\bar{e}^a_{i|j} = \Gamma^k_{ij}\, \bar{e}_k{}^a \mp (1/f)\, \varepsilon^{abc}\, \bar{e}_i{}^b\, \bar{e}_j{}^c\,. \qquad (3.7)$$

But in the case of isometric transformations the Christoffel symbols are invariant because of (3.1)

$$\Gamma^k_{ij}(\bar{\pi}) = \Gamma^k_{ij}(\bar{\pi})\,. \qquad (3.8)$$

Hence, $$\bar{e}_i{}^a(\bar{\pi}) = U^{ab}(\bar{\pi})\, e_i{}^b(\bar{\pi})\,, \qquad (3.9)$$

where U is an orthogonal matrix with $|U| = +1$. Putting (3.9) in (3.7) we conclude that the matrix U has to be independent of π. Consequently the dreibein fields ${}^{(\pm)}e_i^a$ transform linearly under chiral transformations. They are transformed into each other under reflection of the pion coordinates, which is also an isometric transformation of the metric (3.2)

$$\delta\, {}^{(\pm)}e_i^a = \varepsilon^{abc}\, \lambda^b(V)\, {}^{(\pm)}e_i^c\,, \qquad (3.10\,a)$$

$$\delta\, {}^{(\pm)}e_i^a = \pm\, \varepsilon^{abc}\, \lambda^b(A)\, {}^{(\pm)}e_i^c\,, \qquad (3.10\,b)$$

$${}^{(\pm)}e_i^a = R^{ab}\, {}^{(\mp)}e_i^a\,, \qquad (3.10\,c)$$

where $R^{ab} = -\delta^{ab}$ is the reflection matrix. For the following it is sufficient to consider only one set of linearly transforming dreibein fields, e. g. ${}^{(\pm)}e_i^a$ that satisfies (3.6) with the minus sign. We may then drop the superscript $(+)$.

Comparing (2.7) and (3.6) we note for this particular dreibein field

$$C_i^{ab} = -(1/f)\, \varepsilon^{abc}\, e_i^c \qquad (3.11)$$

or according to (3.11)

$$\boldsymbol{C}_\mu(x) = -(1/f)\, \boldsymbol{V}(x)\,. \qquad (3.12)$$

In the gauge (3.12) the field Eqs. (2.12) reduce to the field equations of the $SU(2)$-SUGAWARA model [10]:

$$\partial_\mu \boldsymbol{V}^\mu = 0\,, \qquad (3.13\,a)$$

$$\partial_\nu \boldsymbol{V}_\mu - \partial_\mu \boldsymbol{V}_\nu = -(2/f)\, \boldsymbol{V}_\mu \times \boldsymbol{V}_\nu\,. \qquad (3.13\,b)$$

The solution of (3.13 b) can be written in matrix notation as

$$V_\mu \equiv i\, \boldsymbol{V}_\mu \cdot \boldsymbol{T} = (f/2)\, U^{-1}\, \partial_\mu U\,, \qquad (3.14)$$

where U is an orthogonal matrix and

$$(T^c)^{ab} = i\, \varepsilon^{acb}\,.$$

The parameterization of U determines the coordinates on S_3. The choice

$$U = \exp\{-i(2/f)\, \boldsymbol{\pi}\cdot\boldsymbol{T}\}\,, \qquad (3.15\,a)$$

$$V_\mu = (f/2)\, \exp\{+i(2/f)\, \boldsymbol{\pi}\cdot\boldsymbol{T}\}\, \partial_\mu \exp\{-i(2/f)\, \boldsymbol{\pi}\cdot\boldsymbol{T}\} \qquad (3.15\,b)$$

yields Riemannian normal coordinates. In fact, if we write

$$V_\mu = i\, T^a\, e_i^a(\pi)\, \partial_\mu \pi^i \qquad (3.16)$$

we find

$$e_i^a(\pi) = \left(\frac{\exp\{i(2/f)\, \boldsymbol{\pi}\cdot\boldsymbol{T}\} - 1}{i(2/f)\, \boldsymbol{\pi}\cdot\boldsymbol{T}} \right)^a_i$$

$$= \varepsilon_i{}^a{}_j \frac{\pi^j}{f} \frac{\sin^2 \sqrt{\boldsymbol{\pi}^2/f^2}}{\boldsymbol{\pi}^2/f^2} \qquad (3.17)$$

$$+ \frac{\sin 2\sqrt{\boldsymbol{\pi}^2/f^2}}{2\sqrt{\boldsymbol{\pi}^2/f^2}}\, \delta_a{}^i + \frac{2\sqrt{\boldsymbol{\pi}^2/f^2} - \sin 2\sqrt{\boldsymbol{\pi}^2/f^2}}{2\sqrt{\boldsymbol{\pi}^2/f^2}} \frac{\pi_i\, \pi^a}{\boldsymbol{\pi}^2}\,.$$

It may easily be checked that the dreibein field (3.17) yields the metric (3.2). It satisfies the differential Eq. (3.6) with the minus sign, if we calculate the Christoffel symbols from the metric (3.2). The solution of (3.6) with the plus sign is obtained by parity reflection $\pi^i \to -\pi^i$.

It must be mentioned that the dreibein field (3.17) is closely related to the tangent vectors of the group $SU(2) \times SU(2)$. The tangent vector $t_a{}^i(A)$ for chiral transformations is given by (3.3) while the tangent vector for isospin transformations is

$$t_a{}^i(V) = \varepsilon^i{}_{aj}\, \pi^j\,.$$

Using the metric (3.2) we see that

$${}^{(+)}e_i^a = g_{ij}(1/f)\, \left(t_a{}^j(V) + t_a{}^j(A)\right)\,. \qquad (3.18\,a)$$

Under parity reflection we get

$${}^{(-)}e_i^a = g_{ij}(1/f)\, \left(-t_a{}^j(V) + t_a{}^j(A)\right)\,. \qquad (3.18\,b)$$

It has been shown by BARNES and ISHAM [9] that the dreibein field (3.18) can be constructed out of the tangent vectors.

The field Eqs. (3.13) represent a coordinate-independent formulation of the pion field theory with field variables that transform linearly under chiral transformations. The coordinates are introduced as auxiliary variables by integration of (3.13 b). The situation is quite similar as in electrodynamics, where we have the condition

$$\varepsilon_{\mu\nu\varrho\sigma}\, \partial^\nu F^{\varrho\sigma} = 0$$

instead of (3.13 b). The solution

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

introduces the potentials as auxiliary variables. They have to be calculated from the second field equation

$$\partial^\nu F_{\mu\nu} = 0\,,$$

while we have to calculate the coordinates from (3.13 a) which is equivalent to the field Equation (2.14).

IV. Green's Functions

We now turn to the quantization of the field equations (3.13) in terms of time-ordered Green's functions. To obtain differential equations for the Green's functions we need the equal time commutation relations of the field variables $V_\mu(x)$ [3]

$$[V_0{}^a(x), V_0{}^b(x')]_{x_0=x_0'} = i(2/f)\ \varepsilon^{abc} V_0{}^c(x)\,\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{4.1 a}$$

$$[V_0{}^a(x), V_\alpha{}^b(x')]_{x_0=x_0'} = i\,\delta^{ab}\,\partial_\alpha\,\delta(\boldsymbol{x}-\boldsymbol{x}') + i(2/f)\ \varepsilon^{abc} V_\alpha{}^c(x)\,\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{4.1 b}$$

$$[V_\alpha{}^a(x), V_\beta{}^b(x')]_{x_0=x_0'} = 0\,, \tag{4.1 c}$$

where α and β are spatial Lorentz indices: $\alpha, \beta = 1, 2, 3$. The relations (4.1) can be derived from canonical commutation relations for the pion coordinates

$$[\partial_0 \pi_i(x), \pi^j(x')]_{x_0=x_0'} = -i\,\delta_i{}^j(\boldsymbol{x}-\boldsymbol{x}'), \tag{4.2 a}$$

where

$$\partial_0 \pi_i = g_{ij}\,\partial_0 \pi^j\,. \tag{4.2 b}$$

To obtain (4.1 a) e. g. we use (2.10) and (3.6)

$$[V_0{}^a(x), V_0{}^b(x')]_{x_0=x_0'} = [e^{ai}\,\partial_0 \pi_i(x),\ e^{bj}\,\partial_0 \pi_j(x')]_{x_0=x_0'} = -i(e^{ai}\,e^b_{j|i} - e^{bi}\,e^a_{j|i})\,\partial_0 \pi^j\,\delta(\boldsymbol{x}-\boldsymbol{x}')$$
$$= i(2/f)\ \varepsilon_{bcd}\,e_{cj}\,e_{di}\,e^{ai}\,\partial_0 \pi^j\,\delta(\boldsymbol{x}-\boldsymbol{x}') = i(2/f)\ \varepsilon_{abc} V_0{}^c(x)\,\delta(\boldsymbol{x}-\boldsymbol{x}').$$

The Green's functions can be defined in the usual way. As is well known one has to add extra terms to make the Green's functions covariant, because the commutators (4.1) contain derivatives of deltafunctions. The definition is as follows:

$$G^a_\mu(x) = \langle 0 \,|\, V_\mu{}^a(x) \,|\, 0\rangle, \tag{4.3 a}$$

$$G^{ab}_{\mu\nu}(x,y) = \langle 0 \,|\, T\left(V_\mu{}^a(x)\ V_\nu{}^b(y)\right) \,|\, 0\rangle + i\,g_{\mu 0}\,g_{\nu 0}\,\delta(x-y)\,\delta^{ab}, \tag{4.3 b}$$

$$G^{abc}_{\mu\nu\varrho}(x,y,z) = \langle 0 \,|\, T\left(V_\mu{}^a(x)\ V_\nu{}^b(y)\ V_\varrho{}^c(z)\right) \,|\, 0\rangle + i\,g_{\mu 0}\,g_{\nu 0}\,\delta^{ab}\,\delta(x-y)\,G_\varrho{}^c(z)$$
$$+ i\,g_{\varrho 0}\,g_{\mu 0}\,\delta^{ac}\,\delta(x-z)\,G_\nu{}^b(y) + i\,g_{\varrho 0}\,g_{\nu 0}\,\delta^{bc}\,\delta(y-z)\,G_\mu{}^a(x). \tag{4.3 c}$$

We have introduced $G_\mu{}^a(x)$ to exhibit the structure of the supplementary terms.

The differential equations for the Green's functions follow from the field Eqs. (3.13) and the equal time commutation relations (4.1). For the purpose of illustration we mention three of them:

$$\partial^\mu G^{ab}_{\mu\nu}(x,y) = i\,\delta^{ab}\,\partial_\nu\,\delta(x-y) + i\,\delta(x-y)\,(2/f)\ \varepsilon^{abc} G_\nu{}^c(y), \tag{4.4 a}$$

$$\partial^\mu G^{abc}_{\mu\nu\varrho}(x,y,z) = i\,\delta^{ab}\,\partial_\nu\,\delta(x-y)\ G_\varrho{}^c(z) + i\,\delta^{ac}\,\partial_\varrho\,\delta(x-z)\ G_\nu{}^b(y) \tag{4.4 b}$$
$$+ i\,\delta(x-y)\,(2/f)\ \varepsilon^{abd} G^{dc}_{\nu\varrho}(y,z) + i\,\delta(x-z)\,(2/f)\ \varepsilon^{acd} G^{bd}_{\nu\varrho}(y,z),$$

$$\partial_\varrho G^{ab}_{\mu\nu}(x,y) - \partial_\mu G^{ab}_{\varrho\nu}(x,y) = -(2/f)\ \varepsilon^{acd} G^{cdb}_{\varrho\mu\nu}(x,x,y). \tag{4.4 c}$$

This may be sufficient to derive the equations for the Green's functions in general. To do so it is convenient to use a condensed notation introduced by MANDELSTAM [4]. We consider the totality of functions C_0, $C_\mu{}^a(x_1)$, $C^{ab}_{\mu\nu}(x_1, x_2), \ldots$ as a linear space. The Green's functions represent a single vector of this space (we choose $G_0 = 1$), which we denote by the symbol $|\,G)$, while $|\,C)$ denotes an arbitrary vector. Vectors in the dual space are written in the form $(H\,|$ and are defined as linear forms on $|\,C)$ by means of the scalar product $(H\,|\,C)$. We define the vectors $(H_0\,|$, $(H_\mu{}^a(x_1)|$, $(H^{ab}_{\mu\nu}(x_1, x_2)|$ by means of the equations

$$(H_0\,|\,C) = C_0\,, \qquad (H_\mu{}^a(x_1)\,|\,C) = C_\mu{}^a(x_1), \qquad (H^{ab}_{\mu\nu}(x_1, x_2)\,|\,C) = C^{ab}_{\mu\nu}(x_1, x_2), \tag{4.5}$$
$$\ldots = \ldots\ .$$

Next we introduce a linear operator $\tilde{V}_\mu{}^a(x)$ acting in the dual space:

$$(H_0\,|\,\tilde{V}_\mu{}^a(x) = (H_\mu{}^a(x)|\,, \qquad (H_\nu{}^b(x_1)\,|\,\tilde{V}_\mu{}^a(x) = (H^{ab}_{\mu\nu}(x, x_1)|\,. \quad \ldots = \ldots\ . \tag{4.6}$$

Its action on the vectors $|\,C)$ of the original space can be defined as follows

$$(H_0\,|\,\tilde{V}_\mu{}^a(x)\,|\,C) = (H_\mu{}^a(x)\,|\,C) = (H_0\,|\,\{\tilde{V}_\mu{}^a(x)\,|\,C)\}\ \text{etc.} \tag{4.7}$$

The Green's functions are given by the scalar products of the vectors $(H_0|$, $(H_\mu{}^a(x_1)|\ldots$ and $|G)$:

$$(H_0|G) = G_0 = 1\,, \qquad (H_\mu{}^a(x)|G) = G_\mu{}^a(x)\,, \qquad (H_{\mu\nu}^{ab}(x,y)|G) = G_{\mu\nu}^{ab}(x,y)\,, \tag{4.8}$$

Now we can already condense all rotation field equations like (4.4 c) into the equation

$$\big(\partial_\mu \tilde{V}_\nu{}^a(x) - \partial_\nu \tilde{V}_\mu{}^a(x) - (2/f)\,\varepsilon^{abc}\,\tilde{V}_\mu{}^b(x)\,\tilde{V}_\nu{}^c(x)\big)|G) = 0. \tag{4.9}$$

Taking scalar products with the dual vectors $(H|$, $(H_\mu{}^a(x)|, \ldots$ reproduces the corresponding equations for the Green's functions.

The divergence equations like (4.4 a) and (4.4 b) require the introduction of another linear operator $U^a(x)$ to deal with the terms containing deltafunctions. The action of $U^d(x)$ on the dual space is defined as follows

$$\begin{aligned}(H_{\lambda\mu\nu}^{abc}(x_1, x_2, x_3, \ldots)|\,U^d(x)\\ = i\,\delta^{ad}\,\partial_\lambda\,\delta(x-x_1)\,(H_{\mu\nu}^{bc}(x_2, x_3, \ldots)| + i\,\delta^{bd}\,\partial_\mu\,\delta(x-x_2)\,(H_{\lambda\nu}^{ac}(x_1, x_3, \ldots)| + \ldots\\ + i(2/f)\,\varepsilon_{ade}\,\delta(x-x_1)\,(H_{\lambda\mu\nu}^{ebc}(x, x_1, x_2, \ldots)| + i(2/f)\,\varepsilon_{bde}\,\delta(x-x_2)\,(H_{\lambda\mu\nu}^{aec}(x_1, x, x_3, \ldots)| + \ldots\,.\end{aligned} \tag{4.10}$$

The action of the operator U^d on elements of the original space is defined as in (4.7). From (4.6), (4.7) and (4.10) we can conclude that $U^a(x)$ and $\tilde{V}_\nu{}^b(y)$ obey the commutation relations

$$[U^a(x), \tilde{V}_\nu{}^b(y)] = -i\,\delta^{ab}\,\partial_\nu\,\delta(x-y) - (2/f)\,i\,\varepsilon^{abc}\,\delta(x-y)\,\tilde{V}_\nu{}^c(y). \tag{4.11}$$

The divergence equations can now also be expressed in condensed notation:

$$\big(\partial^\mu \tilde{V}_\mu{}^a(x) - U^a(x)\big)\,|G) = 0\,. \tag{4.12}$$

It is easily checked that scalar multiplication of (4.12) with the dual vectors $(H_0|$, $(H_\mu{}^a(x_1)|\ldots$ yields the Eqs. (4.4 a), (4.4 b) etc., if we observe the commutation relations (4.11) and add the condition

$$(H_0|\,U^a(x) = 0\,. \tag{4.13}$$

In order to obtain a perturbation solution of the field Eqs. (4.9) and (4.12) we first solve Eq. (4.9) in terms of coordinate operators $\tilde{\pi}^i(x)$ $(i = 1, 2, 3)$. The connection between the current operators and the coordinate operators is derived from the connection of the corresponding field variables. Hence for normal coordinates we deduce from (3.16) and (3.17)

$$\tilde{V}_\mu{}^a(x) = \left(\frac{\exp\{i(2/f)\,\tilde{\boldsymbol{\pi}}(x)\cdot\boldsymbol{T}\} - 1}{i(2/f)\,\tilde{\boldsymbol{\pi}}(x)\cdot\boldsymbol{T}}\right)_i^a \partial_\mu\,\tilde{\pi}^i(x) = \sum_{n=1}^{\infty} 1/n!\,\big([i(2/f)\,\tilde{\boldsymbol{\pi}}(x)\cdot\boldsymbol{T}]^{n-1}\big)_i{}^a\,\partial_\mu\,\tilde{\pi}^i(x). \tag{4.14}$$

The introduction of coordinate operators as auxiliary variables suggests the definition of auxiliary dual vectors:

$$\begin{aligned}(H_0|\,, \quad (H^i(x)| = (H_0|\tilde{\pi}^i(x), \qquad (H^{ij}(x,y)| = (H_0|\tilde{\pi}^i(x)\,\tilde{\pi}^j(y),\\ \ldots = \ldots\,.\end{aligned} \tag{4.15}$$

Inserting (4.14) into (4.6) we obtain an expansion of the original dual vectors in terms of auxiliary dual vectors, e. g.,

$$\begin{aligned}(H_\mu{}^a(x)| = (H_0|\,\tilde{V}_\mu{}^a(x) = \sum_{n=1}^{\infty} 1/n!\,\big(i(2/f)\big)^{n-1}\,(H_0|\,\big([\tilde{\pi}(x)\cdot\boldsymbol{T}]^{n-1}\big)_i{}^a\,\partial_\mu\,\partial_\mu\,\tilde{\pi}^{ii}(x)\\ = \sum_{n=1}^{\infty} 1/n!\,\big(i(2/f)\big)^{n-1}\,(T^{i_1}T^{i_2}\ldots T^{i_{n-1}})_{i_n}^{a}\,\partial_\mu{}^{x_n}(H^{i_1\ldots i_n}(x_1, \ldots, x_n)|_{x_1 = x_2 = \ldots = x_n = x}.\end{aligned} \tag{4.16}$$

On the other hand, the coordinate operators are not uniquely determined by (4.14), because we may introduce an arbitrary orthogonal matrix into the definition (3.15 b) of the current in terms of the coordinates:

$$\begin{aligned}V_\mu = (f/2)\,\exp\{i(2/f)\,\boldsymbol{\pi}\cdot\boldsymbol{T}\}\,\partial_\mu\exp\{-i(2/f)\,\boldsymbol{\pi}\cdot\boldsymbol{T}\}\\ = (f/2)\,\exp\{i(2/f)\,\boldsymbol{\pi}\cdot\boldsymbol{T}\}\exp\{i(2/f)\,\boldsymbol{\lambda}\cdot\boldsymbol{T}\}\partial_\mu\,(\exp\{-i(2/f)\,\boldsymbol{\lambda}\cdot\boldsymbol{T}\}\,(\exp\{-i(2/f)\,\boldsymbol{\pi}\cdot\boldsymbol{T}\})\\ = (f/2)\,\exp\{i(2/f)\,\boldsymbol{\pi}'\cdot\boldsymbol{T}\}\,\partial_\mu\exp\{-i(2/f)\,\boldsymbol{\pi}'\cdot\boldsymbol{T}\}\end{aligned} \tag{4.17 a}$$

where $$\exp\{i(2/f)\,\boldsymbol{\pi}(x)\cdot\mathbf{T}\}\,\exp\{i(2/f)\,\boldsymbol{\lambda}\cdot\mathbf{T}\} = \exp\{i(2/f)\,\boldsymbol{\pi}'(x,\lambda)\cdot\mathbf{T}\} \tag{4.17 b}$$

and λ^a are constant real parameters. We now define auxiliary Green's functions in analogy with (4.8):

$$(H_0\,|\,G) = G_0 = 1\,, \qquad (H^i(x)\,|\,G) = G^i(x)\,, \qquad (H^{ij}(x,y)\,|\,G) = G^{ij}(x,y)\,, \tag{4.18}$$

Again these functions are not determined uniquely by the original Green's functions (4.8).

Bearing this fact in mind we want to translate the field Eqs. (4.9) and (4.12) into equations for the auxiliary Green's functions. Equation (4.9) has already been solved by the introduction of the coordinate operators (4.14) and can be dropped. The first term of (4.12) is also easily translated. Writing

$$\widetilde{V}_\mu{}^a(x) = e_i{}^a\big(\tilde{\pi}(x)\big)\,\partial_\mu\tilde{\pi}^i(x)\,, \tag{4.19}$$

where the dreibein field $e_i{}^a$ is given by (3.17), we obtain according to (3.6):

$$\partial^\mu\widetilde{V}_\mu{}^a = e^a{}_i(\tilde{\pi})\,\big(\Box\,\tilde{\pi}^i + \Gamma^i_{jk}\,(\tilde{\pi})\,\partial_\mu\tilde{\pi}^j\,\partial^\mu\tilde{\pi}^k\big)\,. \tag{4.20}$$

To translate the second term of (4.12) we have to introduce a second set of operators $\eta_i(x)$ $(i=1,2,3)$, which we define by their commutation relations with the coordinate operators:

$$[\eta_i(x),\tilde{\pi}^j(y)] = i\,\delta_i{}^j\,\delta(x-y)\,. \tag{4.21}$$

The commutation relations (4.11) are satisfied, if we put

$$U^a(x) = e^{ai}\big(\tilde{\pi}(x)\big)\,\eta_i(x)\,. \tag{4.22}$$

In fact, we obtain by means of (4.21) and (3.6):

$$\begin{aligned}[U^a(x),\widetilde{V}_\nu{}^b(y)] &= [e^{ai}\big(\tilde{\pi}(x)\big)\,\eta_i(x),\ e_j{}^b\big(\tilde{\pi}(y)\big)\,\partial_\nu\tilde{\pi}^j(y)] = e^{ai}\big(\tilde{\pi}(x)\big)\,e_j{}^b\big(\tilde{\pi}(y)\big)\,i\,\delta_i{}^j\,\partial_\nu{}^y\,\delta(x-y)\\ &+ e^{ai}\big(\tilde{\pi}(x)\big)\,e^b_{j|i}\big(\tilde{\pi}(y)\big)\,\partial_\nu\tilde{\pi}^j(y)\,i\,\delta(x-y) = -i\,\delta^{ab}\,\partial_\nu{}^x\,\delta(x-y)\\ &+ [e^{ai}\big(\tilde{\pi}(x)\big)\,e^b_{j|i}\big(\tilde{\pi}(x)\big) - e^{ai}\big(\tilde{\pi}(x)\big)\,e^b_{i|j}\big(\tilde{\pi}(x)\big)]\,\partial_\nu\tilde{\pi}^j(x)\,i\,\delta(x-y)\\ &= -i\,\delta^{ab}\,\partial_\nu{}^x\,\delta(x-y) - (2/f)\,i\,\varepsilon^{abc}\,\widetilde{V}_\nu{}^c(x)\,\delta(x-y)\,.\end{aligned} \tag{4.23}$$

The condition (4.13) is fulfilled, if we write

$$U^a(x) = \eta_i(x)\,e^{ai}\big(\pi(x)\big) \tag{4.24}$$

instead of (4.22) and require

$$(H_0\,|\,\eta_i(x) = 0\,. \tag{4.25}$$

Collecting (4.20) and (4.24) the field Eq. (4.12) is translated as follows

$$(e_i{}^a(\tilde{\pi})\,\big(\Box\tilde{\pi}^i + \Gamma^i_{jk}(\tilde{\pi})\,\partial_\mu\tilde{\pi}^j\,\partial^\mu\tilde{\pi}^k\big) + \eta_i\,e^{ai})\,|\,G) = 0\,, \tag{4.26}$$

or after multiplication with $e_j{}^a$ and summation over a:

$$[g_{ij}(\tilde{\pi})\,\big(\Box\,\tilde{\pi}^i + \Gamma^i_{mn}(\tilde{\pi})\,\partial_\mu\tilde{\pi}^m\,\partial^\mu\,\tilde{\pi}^n\big) + e_j{}^a\,\eta_i\,e^{ai}]\,|\,G) = 0\,. \tag{4.27}$$

Using the commutation relations (4.21) and (3.6) we may rewrite the last term as follows

$$\begin{aligned}e_j{}^a\big(\tilde{\pi}(x)\big)\,\eta_i(x)\,e^{ai}\big(\tilde{\pi}(x)\big) &= \eta_j(x) + i\,\delta(0)\,e_j{}^a\big(\tilde{\pi}(x)\big)\,e^{ai}{}_{|i}\big(\tilde{\pi}(x)\big)\\ &= \eta_j(x) + i\,\delta(0)\,\Gamma^i_{ij}\big(\tilde{\pi}(x)\big) = \eta_j(x) + i\,\delta(0)\,\frac{\partial\ln\sqrt{g}\,\big(\tilde{\pi}(x)\big)}{\partial\,\tilde{\pi}^j(x)}\,,\end{aligned} \tag{4.28}$$

where $g = \det(g_{ij})$. Thus we finally arrive at the following field equation for the auxiliary variables

$$g_{ij}(\tilde{\pi})\left(\Box\tilde{\pi}^i + \Gamma^i_{mn}(\tilde{\pi})\,\partial_\mu\tilde{\pi}^m\,\partial^\mu\tilde{\pi}^n + \eta_j + i\,\delta(0)\,\frac{\partial\ln\sqrt{g}}{\partial\,\pi^j}\right)|\,G) = 0\,. \tag{4.29}$$

The last term is equivalent to the counterterm

$$\Delta\mathcal{L} = -i\,\delta(0)\,\ln\sqrt{g} \tag{4.30}$$

in the Lagrangian that has been discussed by a number of authors [5–7].

Differential equations for the auxiliary Green's functions are obtained from (4.29) by scalar multiplication with the dual vectors (4.15). Having solved these equations by a perturbation series we can construct the coordinate independent Green's functions that transform linearly under chiral transformation according to the prescription (4.14), e. g.

$$G^{ab}_{\mu\nu}(x,y) = \sum_{m=1}^{\infty}\sum_{n=2}^{\infty}\frac{1}{n!\,m!}\left(i\frac{2}{f}\right)^{n+m-2}(T^{i_1}\dots T^{i_{m-1}})^{a}_{i_n}(T^{j_1}\dots T^{j_n})^{b}_{j_n}\;\partial_\mu{}^{x_n}\partial_\nu{}^{y_n} \times G^{i_1\dots i_m,\,j_1\dots j_n}(x_1,\dots,x_m,y_1,\dots,y_n)\,\Big|_{\substack{x_1=\dots=x_n=x\\ y_1=\dots=y_n=y}}\;. \tag{4.31}$$

[1] H. LEHMANN and H. TRUTE, DESY preprint **72,** 8 [1972].
[2] F. KRAUSE and M. SCHEUNERT, BONN preprint [1972].
[3] H. SUGAWARA, Phys. Rev. **170,** 1659 [1968].
[4] S. MANDELSTAM, Phys. Rev. **175,** 1580 and 1604 [1968].
[5] J. HONERKAMP and K. MEETZ, Phys. Rev. D **3,** 1996 [1971].
[6] J. M. CHARAP, Phys. Rev. D **3,** 1998 [1971].
[7] I. S. GERSTEIN, R. JACKIW, B. W. LEE, and S. WEINBERG, Phys. Rev. D **3,** 2486 [1971].
[8] K. MEETZ, J. Math. Phys. **10,** 589 [1969].
[9] K. J. BARNES and C. J. ISHAM, Nucl. Phys. B **15,** 333 [1970].
[10] H. SUGAWARA and M. YOSHIMURA, Phys. Rev. **173,** 1419 [1968].

Generator Coordinate Method and Short Range Correlations *

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To Professor KONRAD BLEULER *on the occasion of his 60th birthday*

It is shown that the cluster expansion formalism previously developed as a basis for a theory of the ground state of correlated systems may be generalised in such a way as to enable the application of the generator coordinate method in the presence of singular interactions, thereby providing also a theory for excited states.

A new interest in the Jastrow method seems to be developing. Investigations of the ground state of correlated Fermi systems, based on cluster expansion formalisms, have been performed by the present authors [1–3], and by others [4–6]. We feel that cluster expansions are also useful for investigating excited states, in particular for extending the generator coordinate method or the random phase approximation (RPA) to systems containing singular interactions. In the present note we describe a method by which the cluster expansion formalism considered in References [2, 3] may be used to apply the generator coordinate method to systems containing short range correlations due to the hard core of the two-body force. In Refs. [1–3] a wave function of the form

$$|\Psi\rangle = \exp S\,|\Phi\rangle \tag{1}$$

where $|\Phi\rangle$ was a Slater determinant,

$$|\Phi\rangle = \prod_{i=1}^{N} a_i^{+}\;|\,0\,\rangle\,, \tag{2}$$

and S was a two-body operator has been considered. In order to describe deviations from equilibrium we will allow S to contain a one-body part. Therefore we write

$$S = \sum_{i=1}^{N} s_i + \tfrac{1}{2}\sum_{i\neq j=1}^{N} f_{ij} = \sum_{mi}(m\,|\,s\,|\,i)\;a_m^{+}\;a_i - \tfrac{1}{2}\sum_{mnij}(m\,n\,|\,f\,|\,i\,j)\;a_m^{+}\,a_n^{+}\;a_j\,a_i\,, \tag{3}$$

where $|\,i\,j)$ and $|\,m\,n)$ are non antisymmetrized states. We use the letters $i, j, k, \dots$ to refer to occupied states (holes), the letters $m, n, p, \dots$ to refer to unoccupied states (particles), and the greek letters $\alpha, \beta, \gamma, \dots$ to refer to either. The quantities $(m\,n\,|\,f\,|\,i\,j)$ are functions of the quantities $(m\,|\,s\,|\,i)$ which may be determined by the prescription that the expectation value of the Hamiltonian H should be a minimum for fixed values of $(m\,|\,s\,|\,i)$. In order to write down cluster expansions we define n-particle uncorrelated states